Order Topology and Others

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Abstract

This thesis is a summary of order topology, ordinal spaces and some exotic topological spaces. All conclusions in it are from the reference books and articles.

Given any set with a linear order, there will be a natural *order topology* on the set. In the first chapter, I'll introduce general properties of linearly ordered topological spaces. In the second chapter I will introduce well-ordered sets, and defined ordinals and ordinal spaces. By introducing some large ordinals, a few topological counterexamples will be given in the third chapter.

Assume that readers have some knowledge of basic set theory and mathematical analysis, so terms like *ordered pair* or *supremum* will not be explained in this thesis. Also, general topology will be briefly reviewed. On that basis, this thesis is consistent, which means every other term will be introduced in a definition or a footnote.

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Chapter 1

Order Topology

1.1 Order

Definition 1.1.1. A *binary relation* between two sets A and B is a subset of the Cartesian product $A \times B$, denoted by R. In common, $\langle x, y\rangle \in R$ is denoted by xRy.

Especially, A binary relation on a set A is a subset of $A \times A$. In other words, it is a collection of ordered pairs of elements of A.

Definition 1.1.2. A *partial order* $\leqslant$ is a binary relation on a set A, which satisfies:

1. $a \leqslant a$;
2. if $a \leqslant b$ and $b \leqslant a$ then $a = b$;
3. if $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$;

$\forall a, b, c \in A$.

Note. *$\leqslant$ is not only a symbol but also a subset of A^2, according to definition 1.1.1.*

Definition 1.1.3. In A, if $a \leqslant b$ and $a \neq b$, we say $a < b$. In other words,

$$< = \leqslant \setminus \{\langle a, a\rangle : \forall a \in A\}$$

We call $<$ a *strict partial order* on A.

Proposition 1.1.1. Strict partial order $<$ has the following properties:

1. $a \nless a$;
2. $a < b$ and $b < a$ can't be true together;
3. if $a < b$ and $b < c$ then $a < c$;

$\forall a, b, c \in A$.

Proof. Trivial. □

Let B be a subset of a partially ordered set $\langle A, \leqslant \rangle$, then there is a partial order inherits naturally.

Definition 1.1.4. The *induced partial order* on B $\leqslant_B$ is defined as $\leqslant \cap B \times B$.

Definition 1.1.5. Two partially ordered sets A and B are called *isomorphic* or having the same *order type*, if there exists a bijection $f : A \to B$, so that

$$a_1 \leqslant a_2 \iff f(a_1) \leqslant f(a_2)$$

The bijection f is called an *isomorphism*.

Definition 1.1.6. A *linear order* is a certain partial order on A, which satisfies: $\forall a, b \in A$, either $a \leqslant b$ or $b \leqslant a$ is true. In other words every two elements are comparable.

The definition of *strict linear order* and *induced linear order* are just like those of partial order, no further introduction.

Given two linear ordered sets A and B, we can define linear orders on their *disjoint union*[1] $A + B$ and their Cartesian product $A \times B$.

Definition 1.1.7. A canonical linear order on $A + B$ can be defined as:

$$\leqslant_{A+B} = \leqslant_A \cup \leqslant_B \cup A \times B$$

That means in addition to their original orders, $a < b, \forall a \in A, b \in B$.

Example 1.1.1. $\mathbb{N}+\mathbb{N}$, here $\mathbb{N}$ is the set of all natural numbers with its standard order. Denote the numbers in the second set with a bar to distinguish them, so we have $0 < 1$, $\bar{0} < \bar{1}$ and $1 < \bar{0}$.

Definition 1.1.8. A canonical linear order on $A \times B$ can be defined as:

$$\langle a_1, b_1 \rangle \leqslant \langle a_2, b_2 \rangle \iff b_1 \leqslant b_2 \text{ or } (b_1 = b_2 \text{ and } a_1 \leqslant a_2)$$

$\forall \langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B$.

Example 1.1.2. $\mathbb{N}\times 2$. We have $\langle 0, 0\rangle < \langle 1, 0\rangle$, $\langle 0, 1\rangle < \langle 1, 1\rangle$ and $\langle 1, 0\rangle < \langle 0, 1\rangle$.

Actually, $\mathbb{N} \times 2$ is isomorphic to $\mathbb{N} + \mathbb{N}$.

In mathematical analysis, we use Dedekind cuts of rational numbers to construct real numbers. In general, a linearly ordered set also has cuts and Dedekind cuts.

Definition 1.1.9. A *cut* of a linearly ordered set A is an ordered pair of nonempty subsets $\langle Z, W \rangle$, satisfying:

1. $Z \cap W = \varnothing$;
2. $Z \cup W = A$;
3. $\forall z \in Z, w \in W, z < w$.

There are four possible types of a cut:

[1] Informally, in disjoint union, even the same elements in different sets are regarded as different elements, in comparison with 'common' union.

1. Z has a greatest element, W has a least element;
2. Z doesn't have a greatest element, W doesn't have a least element;
3. Z has a greatest element, W doesn't have a least element;
4. Z doesn't have a greatest element, W has a least element.

A cut with type (3) or (4) is called a *Dedekind cut*.

1.2 Topology

Definition 1.2.1 (Axioms of Open Sets)**.** A *topology* is a collection of subsets of a set X, denoted by τ and satisfying the following axioms:

1. $\varnothing \in \tau$ and $X \in \tau$.
2. The union of any sets in τ is still in τ.
3. The intersection of any finite number of sets in τ is still in τ.

Sets in τ are called *open sets*. The complement of an open set is called a *closed set*. Together with τ, X is called a *topological space*.

Definition 1.2.2. A collection $\mathfrak{B}$ of some subsets of a set X is called a *base*, if $\overline{\mathfrak{B}}$ is a topology of X, here $\overline{\mathfrak{B}}$ denotes the collection of unions of any elements of $\mathfrak{B}$. We say $\mathfrak{B}$ generates a topology.

Proposition 1.2.1. A given $\mathfrak{B}$ is a base of X if and only if:

1. $\bigcup_{B\in\mathfrak{B}} B = X$;
2. $\forall B_1, B_2 \in \mathfrak{B}, B_1 \cap B_2 \in \overline{\mathfrak{B}}$.

Proof. Necessity is apparent.

Now prove the sufficiency. $\bigcup_{B\in\mathfrak{B}} B = X$ implies axiom of open sets (1), and definition 1.2.2 implies axiom of open sets (2).

To explain how the conditions imply axiom of open sets (3), just select arbitrary U, $U' \in \overline{\mathfrak{B}}$,

$$U = \bigcup_{\alpha} B_\alpha, U' = \bigcup_{\beta} B'_\beta, \text{ here } B_\alpha, B'_\beta \in \overline{\mathfrak{B}}, \forall \alpha, \beta$$

Then

$$U \cap U' = \bigcup_{\alpha,\beta} (B_\alpha \cap B'_\beta)$$

According to condition (2), $B_\alpha \cap B'_\beta \in \overline{\mathfrak{B}}$, $\forall \alpha, \beta$. Then $U \cap U' \in \overline{\mathfrak{B}}$. □

Example 1.2.1. The *Euclidean topology* τ_e on $\mathbb{R}$ is generated by the collection of all open intervals.

Definition 1.2.3. Given a subset Y of a topological space $\langle X, \tau\rangle$, the *subspace topology* τ_Y is defined as $\{Y \cap U : U \in \tau\}$.

Apparently, the definition satisfies the three axioms of open sets. Besides the definition, there is another way to introduce the subspace topology.

Definition 1.2.4. The *subbase* of Y, denoted by $\mathfrak{B}_Y$ is defined as $\{Y \cap B : \forall B \in \mathfrak{B}\}$.

Easy to verify that the subbase generates the subspace topology.

Definition 1.2.5. Given two topological spaces $\langle X, \tau_X \rangle, \langle Y, \tau_Y \rangle$, we use the base $\tau_X \times \tau_Y$ to generate the *product topology* on $X \times Y$.

Some Topological Properties

Now briefly explain some important topological properties which will be mentioned later. Detailed and further properties can be found on Munkres' book *Topology: A First Course*.

Definition 1.2.6. Let X be a topological space. A *neighborhood* of an element x (or a subset) is a subset Y of X that includes an open set U containing x (or the subset). Correspondingly, x is called a *interior point* of Y. The collection of all interior points of Y is called the *interior* of Y.

Definition 1.2.7. Let Y be a subset of X, $x \in X$. If the intersection between $Y \setminus \{x\}$ and every neighborhood of x is nonempty, then x is called a *limit point* of Y. The collection of all limit points of Y is called the *derived set* of Y, denoted by Y'.

Definition 1.2.8 (Separation Axioms)**.**

- T_1: two different elements of X each has a neighborhood not containing the other element.
- T_2: two different elements of X have disjoint neighborhoods.
- T_3: a closed subset of X and an element not in it have disjoint neighborhoods.
- T_4: two disjoint closed sets of X have disjoint neighborhoods.
- *completely normal*: every subspace of X is T_4 $\iff$ every two *separated sets*[2] can be separated by neighborhoods.
- T_5: T_1 and completely normal.

Proposition 1.2.2. T_2 implies T_1; T_1 along with T_4 implies T_2 and T_3.

Proof. The first statement is trivial.

T_1 is equivalent to that every *singleton*[3] is closed, which is quite easy to verify. Then the second statement becomes apparent. □

Definition 1.2.9. X is *connected* if X can't be the union of two disjoint nonempty open subsets.

[2] Each is disjoint from the other's closure (A's closure $= A \cup A'$).

[3] A set which contains only one element.

Definition 1.2.10. X is *path-connected* if there exists a path joining any two elements in X, i.e. $\forall x, y \in X$, there exists a continuous function $f : [0,1] \to X, f(0) = x, f(1) = y$.

Definition 1.2.11. X is *compact* if every open cover of X has a finite subcover.

Definition 1.2.12. X is *sequentially compact* if every sequence of X has a converging subsequence.

Definition 1.2.13 (Countability Axioms)**.**

- C_1: each element of X has a countable neighborhood basis ($\forall U, \exists B \subseteq U$, here U is a neighborhood of x and B is in the neighborhood basis).
- C_2: X has a countable base.

Apparently, C_2 implies C_1.

1.3 Order Topology

In $\mathbb{R}$, we can choose the collection of all open intervals as a base to generate the Euclidean topology. Similarly, we can generate the order topology on any linearly ordered set A. We define open intervals on A as these:

- $(a, b) = \{c : a < c < b\}$;
- $(a, \to) = \{c : a < c\}$;
- $(\leftarrow, b) = \{c : c < b\}$;

$\forall a, b, c \in A$.

Definition 1.3.1. These kinds of sets are called *open intervals* on A.

Definition 1.3.2. The *order topology* on a linearly ordered set containing more than one element is generated by the collection of all open intervals. A space whose topology is an order topology is called a *linearly ordered topological space (LOTS).*

The definition is proper, because it's quite easy to verify that the collection of all open intervals satisfies both conditions in proposition 1.2.1.

Note. *Whenever a LOTS is mentioned, we always assume it contains more than one element to avoid being loaded down with trivial details.*

The next two propositions show that any LOTS is T_5.

Proposition 1.3.1. Any LOTS is T_2.

Proof. Let X denote a LOTS. $\forall a, b \in X, a \neq b$, either $a < b$ or $b < a$ due to definition 1.1.3 and 1.1.6. Without loss of generality, we assume that $a < b$.

If there exists a c so that $a < c < b$ then $(\leftarrow, c)$ and $(c, \to)$ are the required neighborhoods. If not, namely $(a, \to) \cap (\leftarrow, b) = (a, b) = \varnothing$, therefore $(\leftarrow, b)$ and $(a, \to)$ are disjoint neighborhoods of a and b. □

Proposition 1.3.2. Any LOTS is completely normal.

Proof. Let A and B be separated subsets of X. For each $a \in A$ there is an open interval U_a such that $a \in U_a \subseteq X \setminus B$, and for each $a \in B$ there is an open interval U_b such that $b \in U_b \subseteq X \setminus A$. Let $U_A = \bigcup_{a \in A} U_a$ and $U_B = \bigcup_{b \in B} U_a$; clearly $A \subseteq U_A$, $B \subseteq U_B$, and $U_A \cap U_B \subseteq X \setminus (A \cup B)$.

Let $U = U_A \cap U_B$. If $U = \varnothing$, then the proof is done, so assume that $U \neq \varnothing$. Define a relation $\sim$ on U: $p \sim q \iff [\min\{p, q\}, \max\{p, q\}] \subseteq U$; it's easily verified that $\sim$ is an equivalence relation.

Let $T \subseteq U$ contain exactly one point of each $\sim$-class.[4] If $a \in A$, and $p, q \in U_a \cap T$ with $p < q$, now prove that $p < a < q$. Suppose that $a < p$. Since $p \in T \subseteq U$, there is a $b \in B$ such that $p \in U_b$; $[a, q] \subseteq U_a \subseteq X \setminus B$, so $b \notin [a, q]$. If $b < a$, then $a \in [b, p] \subseteq U_b \cap A = \varnothing$, so $a < p < q < b$. But then $[p, q] \subseteq U_a \cap U_b \subseteq U$, so $p \sim q$, contradicting the choice of p and q, therefore $p < a$. Similarly $a < q$. Similarly, if $a \in B$ and $p, q \in U_a \cap T$ with $p < q$, then $p < a < q$. Therefore $|U_a \cap T| \leqslant 2$, $\forall a \in A \cup B$.

Now fix $p \in T$. Let $A_p = \{a \in A : p \in U_a\}$ and $B_p = \{a \in B : p \in U_a\}$, $A_p \neq \varnothing \neq B_p$ since $p \in U$. Suppose that $a < p$ for some $a \in A_p$. If $b \in B_p$ and $b < p$, then either $a < b$ and $b \in U_a$, or $b < a$ and $a \in U_b$, since the sets U_a and U_b are open intervals. So $p < b$, and since $b \in B_p$ was arbitrary, $p < B_p$. Similarly $A_p < p$ and therefore $A_p < p < B_p$. If instead $p < a$ for some $a \in A_p$, it shows similarly that $B_p < p < A_p$.

$\forall a \in A \cup B$, define V_a as these:

$$V_a = \begin{cases} U_a, & \text{if } U_a \cap T = \varnothing \\ U_a \cap (p, \rightarrow), & \text{if } U_a \cap T = \{p\} \text{ and } p < a \\ U_a \cap (\leftarrow, p), & \text{if } U_a \cap T = \{p\} \text{ and } a < p \\ U_a \cap (p, q), & \text{if } U_a \cap T = \{p, q\} \text{ and } p < a < q \end{cases}$$

Let $V_A = \bigcup_{a \in A} V_a, V_B = \bigcup_{a \in B} V_a$, apparently V_A and V_B are open, $A \subseteq V_A, B \subseteq V_B$, now prove that $V_A \cap V_B = \varnothing$.

Suppose not, then there exists $a \in A$ and $b \in B$ such that $V_a \cap V_b \neq \varnothing$; without loss of generality assume that $a < b$. Fix $q \in V_a \cap V_b$, it's easy to see that $a < q < b$, since V_a and V_b are open intervals. Moreover, $q \in U$, so $q \sim p$ for a unique $p \in T$. Let $I = [\min\{p, q\}, \max\{p, q\}]$. Then $I \subseteq U \subseteq X \setminus (A \cup B)$, so $a, b \notin I$, and therefore $a < p < b$. If $p \leqslant q$, then $p \in V_a \cap T \subseteq U_a \cap T$, and by construction $V_a \subseteq (\leftarrow, p)$, and $p \notin V_a$, that leads to a contradiction. If $q \leqslant p$, then $p \in V_b \cap T \subseteq U_b \cap T$, so that $V_b \subseteq (p, \rightarrow)$, and $p \notin V_b$, which is again a contradiction. Therefore $V_A \cap V_B = \varnothing$, X is completely normal. □

Corollary. Any LOTS is T_5.

Consider a subset Y of X. There are two topologies on Y: the induced order topology $\tau_{\leqslant_Y}$ and the subspace topology τ_Y. Sometimes they are the same: $\mathbb{Q}$ as a subset of $\langle \mathbb{R}, \leqslant, \tau \rangle$; sometimes not:

Example 1.3.1. If $Y = \{-1\} \cup \{\frac{1}{n} : n \in \mathbb{N}_+\}$ as a subset of $\langle \mathbb{R}, \leqslant, \tau \rangle$, then $\{-1\}$ is open in τ_Y, but not open in $\tau_{\leqslant_Y}$.

In fact, τ_Y is always thinner than $\tau_{\leqslant_Y}$, which means $\tau_{\leqslant_Y} \subseteq \tau_Y$. Just note that every set in the base of $\tau_{\leqslant_Y}$ is also in the base of τ_Y. But when are they the same? We have the following conclusion.

[4] The axiom of choice is required here.

Proposition 1.3.3. $\tau_{\leqslant_Y} = \tau_Y \iff$ for every Dedekind cut $\langle Z, W\rangle$ of Y, if Z has a greatest element z, then z is the infimum of W; if W has a least element w, then w is the supremum of Z.

Proof. Let's start with necessity. Suppose there exists a Dedekind cut $\langle Z, W\rangle$ of Y, without loss of generality, assume that Z has a greatest element z while W doesn't have a least element, and z isn't the infimum of W. So $\exists x \in X, z < x$ and $x < w, \forall w \in W$. Therefore $Z = (\leftarrow, x) \cap Y$, is an element of τ_Y. However, since W doesn't have a least element, Z can't be an element of $\tau_{\leqslant_Y}$.

Now prove the sufficiency. We only need to prove: $\tau_Y \subseteq \tau_{\leqslant_Y}$, which is equivalent to that the intersection between any open interval of X and Y is an open set of $\tau_{\leqslant_Y}$.

According to axiom of open sets (3), a bounded open interval is the intersection between two unbounded open intervals, so only unbounded open intervals need to be verified: $\forall x \in X$, $(\leftarrow, x) \cap Y$ and $(x, \rightarrow) \cap Y$ are elements of $\tau_{\leqslant_Y}$.

If $x \in Y$, then $(\leftarrow, x) \cap Y$ and $(x, \rightarrow) \cap Y$ are also open intervals in $\langle Y, \leqslant_Y\rangle$. If $x \notin Y$, $\langle Z, W\rangle = \langle (\leftarrow, x) \cap Y, (x, \rightarrow) \cap Y\rangle$ is a cut of Y (assume Z and W are both nonempty). Due to the condition, the cut can't be a Dedekind cut, otherwise x must be in Y. If Z has a greatest element z and W has a least element w, then

$$Z = (\leftarrow, x) \cap Y = (\leftarrow, w)_Y$$

$$W = (x, \rightarrow) \cap Y = (z, \rightarrow)_Y$$

are elements of $\tau_{\leqslant_Y}$. If Z doesn't have a greatest element and W doesn't have a least element, then

$$Z = (\leftarrow, x) \cap Y = \bigcup_{z \in Z} (\leftarrow, z)_Y$$

$$W = (x, \rightarrow) \cap Y = \bigcup_{w \in W} (w, \rightarrow)_Y$$

are also elements of $\tau_{\leqslant_Y}$. □

Chapter 2

Ordinal Space

2.1 Well-Order

Definition 2.1.1. A *well-order* on a set A is a certain linear order, satisfying that every nonempty subset of A has a least element.

Example 2.1.1. Any finite linearly ordered set is well-ordered.

Example 2.1.2. $\mathbb{N}$ is well-ordered with its standard order.

Example 2.1.3. $\mathbb{N} \times \mathbb{N}$ is well-ordered with its canonical order.

Proof. For a nonempty subset $\{\langle m, n\rangle : m, n \in \mathbb{N}\}$, firstly find the least n in the subset, then find the least m in with the least n. Then we get the least element. We can do that because $\mathbb{N}$ is well-ordered. □

Corollary. For any finite number of well-ordered sets $A_1, A_2, \ldots, A_n$, $A_1 \times A_2 \times \cdots \times A_n$ is well-ordered with its canonical order.

Proposition 2.1.1. Any element a except the greatest element of a well-ordered set A has a *successor* b, meaning $a < b$ and there doesn't exist c satisfying $a < c < b$.

However an element isn't necessarily a successor of another element.

Proof. Since a isn't the greatest element, $\{x > a : x \in A\}$ is nonempty so there exists a least element b which is the successor.

Consider the well-ordered set $\mathbb{N} \times \mathbb{N}$, the element $\langle 0, 1\rangle$ isn't a successor of any other element. □

If an element isn't a successor of any other element, then we call it a *limit element*. The least element is also a limit element. Correspondingly, an element is called a *predecessor* of its successor.

Proposition 2.1.2. A is well-ordered $\iff$ there doesn't exist $a_i \in A, \forall i \in \mathbb{N}$, so that $a_0 > a_1 > a_2 > \cdots$

Proof. If there exists such a_i, so $\{a_i : \forall i \in \mathbb{N}\}$ doesn't has a least element then A is not well-ordered. Thus the sufficiency is proved.

If A isn't well-ordered then there exists a subset B which doesn't a least element. Select an arbitrary element b_0 in B. Due to the hypothesis, b_0 isn't the least element. So there exists $b_1 < b_0$. In the same way, there exists $b_2 < b_1, \ldots$, then we have the infinite descending $\{b_i, \forall i \in \mathbb{N}\}$. Thus the necessity is proved. $\square$

Proposition 2.1.3 (Transfinite Induction)**.** Let A be a well-ordered set, $p(a)$ be a proposition about element a. For any element $a \in A$, if $\forall b < a, p(b)$ is true $\Longrightarrow p(a)$ is true, then $\forall a \in A$, $p(a)$ is true.

Proof. Let B be the collection of all elements that doesn't satisfy p. Suppose B is nonempty, then B has a least element b. However for all elements less than b, p is true, so $p(b)$ must be true. Therefore $B = \varnothing$, $\forall a \in A, p(a)$ is true. $\square$

Note. *The condition already implies* $p(0)$ *is true, here* 0 *denotes the least element.*

Definition 2.1.2. Let $\langle Z, W \rangle$ be a cut of a well-ordered set A, we call Z an *initial segment* of A. $\varnothing$ and A itself are also called initial segments.

Proposition 2.1.4. For any two well-ordered sets A and B, either A is isomorphic to an initial segment of B, or B is isomorphic to an initial segment of A.

Proof. Define a function $f : A \to B, a \mapsto$ the least element in B that doesn't equal $f(a'), \forall a' < a$. If $\{f(a') : \forall a' < a\} = B$, then $f(a)$ will not be defined. This definition ensures f to be strictly increasing.

If f is defined on the whole set A, then the image of f must be an initial segment of B. That is because $\forall b < f(a), b$ must be in the image of f according to the definition of $f(a)$. Therefore f is an isomorphism between A and an initial segment of B.

If f is defined on an initial segment of A ($\neq A$), then the image of f must equal B, according to the definition. Therefore f is an isomorphism between an initial segment of A and B. $\square$

Proposition 2.1.5. *A can't be isomorphic to an initial segment of it, unless the initial segment is A itself.*

Proof. Suppose A is isomorphic to $[0, a), a \in A$, let f denote the isomorphism. So $\exists a' \in A, f(a') < a'$. Because f is strictly increasing, we get

$$a' > f(a') > f(f(a')) > f(f(f(a'))) > \cdots$$

That contradicts proposition 2.1.2. $\square$

Corollary. If A is isomorphic to an initial segment of B and B is isomorphic to an initial segment of A, then $A = B$.

Proof. Suppose $A \neq B$ and the two statements are both true, we can get a initial segment ($\neq A$) of A that is isomorphic to A, using the two isomorphisms. That leads to a contradiction. $\square$

These propositions allow us to compare two well-ordered sets. For any well-ordered sets A and B, only one of these is true:

- if A is isomorphic to an initial segment of B ($\neq B$), then we say A's order type is less than B's order type;

- if they are isomorphic, then we say A's order type is equal to B's order type;
- if an initial segment of A ($\neq A$) is isomorphic to$B$, then we say A's order type is greater than B's order type.

2.2 Ordinal

As written before, two isomorphic ordered sets are said to have the same order type. Informally speaking, an ordinal is the order type of all isomorphic well-ordered sets.

However the collection of all isomorphic well-ordered sets may be too 'big' to be a set. To avoid the difficulty, we can choose a standard set from every collection of isomorphic well-ordered sets as the ordinal. So we adopt von Neumann's definition:

Definition 2.2.1. An *ordinal* is the collection of all ordinals less than it; $0 = \varnothing$ is the least ordinal.

More strictly, a set α is an ordinal if and only if:

1. the inclusion relation $\in$ on α is a well-order;
2. any element of α is also a subset of α.

For example, $1 = \{\varnothing\}$ is the successor of 0, $2 = \{\varnothing, \{\varnothing\}\}$ is the successor of 1, and $\omega = \{0, 1, 2, \ldots\}$ is the least infinite ordinal. $\mathbb{N}$ with its standard order has the ordinal ω.

With no doubt, an ordinal is a well-ordered set, so we can compare two ordinals. If α is isomorphic (actually equal) to an initial segment of β, then we say $\alpha \leqslant \beta$.

If $\alpha < \beta$, then both $\alpha \in \beta$ and $\alpha \subsetneqq \beta$ are true. In fact,

Proposition 2.2.1. $\alpha < \beta \iff \alpha \in \beta \iff \alpha \subsetneqq \beta$

Proof. That is easy to see, just give some consideration to the definition of ordinals. □

A notable fact is, the collection of all ordinals is not a ordinal, not even a set. If it was, then it must be the greatest ordinal, but any ordinal has a successor which is greater than itself. This conclusion is known as the Burali-Forti paradox.

Ordinal Arithmetic

Since the elements of an ordinal are all ordinals, we can say that an ordinal is the successor of another ordinal. Or we can redefine the term 'successor' in another way, due to the specificity of ordinals.

Definition 2.2.2. The *successor* of an ordinal α is defined as $\alpha \cup \{\alpha\}$, denoted by $\alpha+1$. Apparently, $\alpha+1$ is also an ordinal. An ordinal is called the *predecessor* of its successor. This definition is consistent with the previous definition.

Proof. $\alpha \in \alpha + 1$, so $\alpha < \alpha + 1$. Suppose $\exists \beta, \alpha < \beta < \alpha + 1$, then $\beta \in \alpha \cup \{\alpha\}$, so either $\beta \in \alpha \iff \beta < \alpha$ or $\beta \in \{\alpha\} \iff \beta = \alpha$. That leads to a contradiction. Therefore the two definitions are consistent. □

An ordinal is called a *successor ordinal*, if it has a predecessor. Else called a *limit ordinal*.

Proposition 2.2.2. A successor ordinal has a predecessor, which is precisely its greatest element in it.

Proof. Let $\alpha + 1$ denote a successor ordinal. If $\exists \beta \in \alpha + 1$ which is greater than α, then $\alpha < \beta < \alpha + 1$ so $\alpha + 1$ can't be the successor of α. Therefore α is the greatest element of $\alpha + 1$.

If an ordinal α has a greatest element β, then $\alpha > \beta$. If $\alpha > \beta + 1$, then β can't be the greatest element because $\beta + 1$ is also in α. So $\beta < \alpha \leqslant \beta + 1$. There doesn't exist an ordinal between β and $\beta + 1$, so $\alpha = \beta + 1$. That means β is the predecessor of α, making α a successor ordinal. □

Proposition 2.2.3. If $\forall i \in I$ (here I is an index set), α_i is an ordinal, then $\bigcup_i \alpha_i$ is also an ordinal, and it's precisely the supremum of $\{\alpha_i\}$.

Proof. To prove that $\bigcup_i \alpha_i$ is an ordinal, we only need to prove that if $\alpha \in \bigcup_i \alpha_i$ then $\beta \in \bigcup_i \alpha_i$ for any $\beta < \alpha$. That is quite apparent because if α is in some α_i then β is in that α_i too, for any $\beta < \alpha$.

$\bigcup_i \alpha_i$ is a upper bound of $\{\alpha_i\}$ because $\forall i, \alpha_i < \bigcup_i \alpha_i$. And every upper bound of $\{\alpha_i\}$ must contain $\bigcup_i \alpha_i$ as a subset according to proposition 2.2.1. Therefore $\bigcup_i \alpha_i$ is the least upper bound, namely the supremum of $\{\alpha_i\}$. □

After those preparatory work, we can finally define the sum of ordinals. We have defined the canonical order on $A + B$ in section 1.1 , here A and B are both linearly ordered sets. Naturally, we want the sum of two ordinals $\alpha + \beta$ to be the order type of $A + B$, here A and B each has the order type α and β.

Definition 2.2.3 (Addition of Ordinals)**.** Since we have known that $\alpha + 1 = \alpha \cup \{\alpha\}$, we can define $\alpha + \beta$ for any ordinal β recursively:

1. $\alpha + 0 = \alpha$;
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$;
3. $\alpha + \beta = \bigcup_{\gamma < \beta} (\alpha + \gamma)$, for limit ordinal $\beta \neq 0$.

Proposition 2.2.4. If A and B each has the order type α and β, then $A + B$ has the order type $\alpha + \beta$.

Proof. We use the transfinite induction to prove it. Assume $\forall \gamma < \beta$, $A + C$ has the order type $\alpha + \gamma$, here γ is the order type of C.

If $\beta = 0$ then $B = \varnothing$, the proposition becomes trivial.

If $\beta = \gamma + 1$ is a successor then B has a greatest element b, $B = B \setminus \{b\} \cup \{b\}$. According to the hypothesis, there is an isomorphism between $\alpha + \gamma$ and $A + (B \setminus \{b\})$. In addition, by mapping γ(as an element of β) to b we can get an isomorphism between $\alpha + \beta$ and $A + B$.

If β is a limit ordinal, then for any initial segment C of B ($C \neq B$), we have an isomorphism between $\alpha + \gamma$ and $A + C$, here γ is the order type of C. All the isomorphisms are consistent so we can paste them together, getting an isomorphism between $\alpha + \beta$ and $A + B$. □

Example 2.2.1. $0 + \alpha = \alpha, 1 + \omega = 2 + \omega = \omega, \omega + 1 \neq \omega$.

We can find out two facts by this example: firstly, the addition of two ordinals isn't commutative; secondly, $\alpha + \gamma$ may equal $\beta + \gamma$ even if $\alpha \neq \beta$, so the subtraction of two ordinals doesn't always make sense.

We can use the addition of ordinals to define the multiplication:

Definition 2.2.4 (Multiplication of Ordinals)**.**

1. $\alpha \cdot 0 = 0$;
2. $\alpha(\beta + 1) = \alpha\beta + \alpha$;
3. $\alpha\beta = \bigcup_{\gamma<\beta}(\alpha\gamma)$, for limit ordinal $\beta \neq 0$.

Similar to proposition 2.2.4 both in conclusion and in proof, if A and B each has the order type α and β, then $A \times B$ has the order type $\alpha\beta$.

Example 2.2.2. $0\alpha = 0, 2\omega = 3\omega = \omega, \omega \cdot 2 = \omega + \omega$.

Similarly too, the multiplication of two ordinals isn't commutative and the division of two ordinals doesn't always make sense.

Finally, we use the multiplication to define the exponentiation:

Definition 2.2.5 (Exponentiation of Ordinals)**.**

1. $\alpha^0 = 1$;
2. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$;
3. $\alpha^\beta = \bigcup_{\gamma<\beta}(\alpha^\gamma)$, for limit ordinal $\beta \neq 0$.

In the three definitions of ordinal arithmetic, we have only used two basic steps to create new objects: getting a successor of an ordinal and getting the union of some ordinals. According to definition 2.2.2 and proposition 2.2.3, we always get an ordinal after every step.

2.3 Ordinal Space

After spending so much space in set theory, let's talk about something more 'topological': ordinals as linearly ordered topological spaces. As a LOTS, an ordinal α is often written as $[0, \alpha)$ to emphasize that it's the space consisting of all ordinals less than α, and $\alpha + 1$ is written as $[0, \alpha]$.

Note. *When 'the topology' of an ordinal is mentioned, it always refers to the order topology.*

Proposition 2.3.1. Any ordinal space is *totally disconnected*[1].

Proof. Let S be a subset of an ordinal and S has more than one element, then S has a least element β. Both $(\leftarrow, \beta + 1) \cap S = \{\beta\}$ and $(\beta, \rightarrow) \cap S = S \setminus \{\beta\}$ are nonempty and open in S, so S is disconnected. Therefore the ordinal space is totally disconnected. □

[1] Any subset having two element or more is disconnected.

For any finite ordinal, the topology is simply the *discrete topology*[2], so only infinite ordinals can be interesting.

Successor ordinals less than α are *isolated*[3] in α. That is because $\{\beta+1\} = (\beta, \beta+2)$. Meanwhile, limit ordinals (except 0) less than α are not isolated in α. Just note that there doesn't exist a greatest element less than a limit ordinal, so any open interval containing the limit ordinal must contain some ordinals less than it. Therefore the derived set of α is precisely the set consisting of limit ordinals (except 0) less than α.

Proposition 2.3.2. $[0, \alpha)$ is compact $\iff$ α is a successor ordinal.

Proof. If α is a limit ordinal, then the open cover $\{[0, \beta)\}_{\beta<\alpha}$ can't have a finite subcover because any finite subcover has a greatest element (elements of the cover are ordinals) and the greatest element is less than α.

If α is a successor ordinal, then $[0, \alpha)$ has a greatest element α_0. Let $\mathfrak{S}$ be a open cover of $[0, \alpha)$. Choose a open set S_0 in the cover that contains α_0, then there is a open interval $I_0 \subseteq S_0$ that contains α_0. The open interval I_0 looks like $(\alpha_1, \rightarrow)$, so $[0, \alpha) \setminus I_0$ also has a greatest element α_1. Similarly, choose S_1 (S_1 may equal S_0 but that doesn't matter) and I_1 and find the greatest element of $[0, \alpha) \setminus (I_0 \cup I_1)$, denoted by α_2 ... According to proposition 2.1.2, this process must end after finite steps. Then $\{I_i\}$ is a finite cover of $[0, \alpha)$, therefore $\{S_i\}_{i=1,\ldots,n}$ is a finite subcover of $\mathfrak{S}$. □

Example 2.3.1. $[0, \omega)$, $[0, \omega]$

Let's talk about the least two transfinite ordinals.

ω is the ordinal of $\mathbb{N}$ with its standard order. ω looks like $\{0, 1, 2, 3, \ldots\}$. The topology is also the discrete topology.

$\omega + 1$ is the successor of ω, and it looks like $\{0, 1, 2, 3, \ldots \omega\}$. The topology is not the discrete topology, because $\{\omega\}$ is not open.

$[0, \omega]$ is compact while $[0, \omega)$ isn't. $[0, \omega]$ is the one-point compactification (adding a greatest element to get a compact space) of $[0, \omega)$. In common, for any limit ordinal $\alpha \neq 0$, $[0, \alpha]$ is the one-point compactification of $[0, \alpha)$.

Example 2.3.2. $[0, \omega_1)$, $[0, \omega_1]$

Let ω_1 denote for the least uncountable ordinal. From Zermelo's Theorem[4] we know that there does exist an uncountable ordinal. The least uncountable ordinal ω_1 consists of all countable or finite ordinals.

Proposition 2.3.3. $[0, \omega_1)$ is sequentially compact.

Proof. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of $[0, \omega_1)$. We can get an increasing subsequence by mapping n to the n-th least element of $\{\alpha_n\}$. Then the supremum (or union) of that subsequence is also countable so it's an element of $[0, \omega_1)$. The supremum couldn't be a successor ordinal, otherwise its predecessor is a upper bound too. If an open neighborhood of the supremum (must contains another element because the supremum is a limit ordinal) doesn't contain any element in $\{\alpha_n\}$, then the least element of the neighborhood is also an upper bound. Therefore the supremum is a limit point of $\{\alpha_n\}$, we have a converging subsequence of $\{\alpha_n\}$. □

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[2] With discrete topology, any subset of the space is open.

[3] x is isolated in X means $\{x\}$ is open in X.

[4] Every set can be well-ordered.

According to proposition 2.3.2, $[0, \omega_1)$ is not compact. As a result, $[0, \omega_1)$ is not metrizable. And a more interesting fact is,

Proposition 2.3.4. In $[0, \omega_1]$, ω_1 is a limit point of $[0, \omega_1)$, but any sequence of $[0, \omega_1)$ couldn't converge to ω_1.

Proof. ω_1 is a limit point of $[0, \omega_1)$ since $\{\omega_1\}$ is not open, and the second statement is because the supremum (=union) of any countable subset of $[0, \omega_1)$ (elements are countable ordinals) is a countable ordinal. □

As a corollary, $[0, \omega_1]$ is not C_1 since the only element without a countable neighborhood basis is ω_1. That shows the subspace $[0, \omega_1)$ is C_1.

Proposition 2.3.5. Neither $[0, \omega_1)$ nor $[0, \omega_1]$ is C_2.

Proof. We only need to prove that $[0, \omega_1)$ isn't C_2, since $[0, \omega_1]$ is not even C_1. Suppose $[0, \omega_1)$ is C_2, then it has a countable base. Let α be the supremum of the collection of the least elements from each set in that base. Because the base is countable, so $\alpha \in [0, \omega_1)$. Then $(\alpha, \rightarrow)$ is not open because any open set must contain an ordinal less than or equal to α, according to the construction of α. That leads to a contradiction. □

Chapter 3

Counterexamples

3.1 Long Ray

The ray $[0, +\infty)$ can be defined as $[0, 1) \times \omega$, as a product of two linearly ordered sets. With this definition, $\forall a \in \{0\} \cup \mathbb{R}_+, a = \langle a - \lfloor a \rfloor, \lfloor a \rfloor \rangle$, here $\lfloor \cdot \rfloor$ is the floor function.

Now replace ω by ω_1, we get:

Definition 3.1.1. The *long ray* $L = [0, 1) \times \omega_1$.

Proposition 3.1.1. *L is sequentially compact.*

Proof. Let $\{\langle x_n, \alpha_n \rangle\}_{n \in \mathbb{N}}$ be a sequence of $[0, \omega_1)$. Since $[0, \omega_1)$ is sequentially compact, there is a converging subsequence of $\{\alpha_n\}$, denoted by $\{\alpha_i\}_{i \in \mathbb{N}}$, converges to α. $[0, 1]$ is also sequentially compact so $\{x_i\}$ has a converging subsequence, denoted by $\{x_j\}_{j \in \mathbb{N}}$, converges to $x \in [0, 1]$. Therefore $\{\langle x_j, \alpha_j \rangle\}$ converges to $\langle x, \alpha \rangle$ (if $x = 1$ then $\langle 1, \alpha \rangle = \langle 0, \alpha + 1 \rangle$). $\square$

Consequently, L isn't *homeomorphic*[1] to $[0, 1)$ since $[0, 1)$ is not sequentially compact.

Proposition 3.1.2. *L is path-connected and is therefore connected.*

We need a lemma to prove this proposition.

Lemma. *For any ordinal $\alpha \in (0, \omega_1)$, $[0, 1) \times \alpha$ is homeomorphic to $[0, 1)$, and the homeomorphism can be chosen to be order-preserving.*

Proof. First of all, $[0, 1) \times 1 = [0, 1)$. Assume $\forall \beta \in (0, \alpha)$, $[0, 1) \times \beta$ is order-preservingly homeomorphic to $[0, 1)$.

If $\alpha = \beta + 1$ is a successor ordinal, then we have two order-preserving homeomorphisms: $f_1 : [0, 1) \times \beta \to [0, \frac{1}{2}), f_2 : [0, 1) \times \{\beta\} \to [\frac{1}{2}, 1)$. By pasting f_1 and f_2 together we can get a homeomorphism between $[0, 1) \times \alpha$ and $[0, 1)$.

If α is a limit ordinal, we will find an increasing sequence $\alpha_0 < \alpha_1 < \alpha_2 < \cdots$ that converges to α ($\forall \beta < \alpha, \exists i$, so that $\alpha_i > \beta$). Because $(0, \alpha)$ is countable, we can list the ordinals as $\gamma_0, \gamma_1, \gamma_2, \ldots$, in this list the ordinals are not necessarily

[1] There exists a bijection between two homeomorphic spaces, preserving all topological properties. The bijection is called a *homeomorphism*.

ordered. Now let $\alpha_0 = \gamma_0$. If $i > 0$, let α_i be γ_j, here j is the least index so that $\gamma_j > \alpha_{i-1}$. Such α_i does exist because α is a limit ordinal, and α_i will be greater than any certain ordinal less than α because the γ-list contains all ordinals less than α.

For any i, there exists a homeomorphism which maps $[0,1) \times \alpha_{i+1}$ to $[0,1)$. This homeomorphism maps $\langle 0, \alpha_i \rangle$ to $A \in [0,1)$, so it maps $[\langle 0, \alpha_i \rangle, \langle 0, \alpha_{i+1} \rangle)$ to $[A, 1)$. Rescale these homeomorphisms so $[\langle 0, 0 \rangle, \langle 0, \alpha_0 \rangle)$ maps to $[0, \frac{1}{2})$, $[\langle 0, \alpha_0 \rangle, \langle 0, \alpha_1 \rangle)$ maps to $[\frac{1}{2}, \frac{3}{4})$ and $[\langle 0, \alpha_1 \rangle, \langle 0, \alpha_2 \rangle)$ maps to $[\frac{3}{4}, \frac{7}{8})$, etc. Paste them together and we get an order-preserving homeomorphism between $[0,1) \times \alpha$ and $[0,1)$. □

Now we can prove proposition 3.1.2.

Proof. According to the lemma, for any ordinal $\alpha \in (0, \omega_1)$, $[0,1) \times \alpha$ is path-connected because $[0,1)$ is path-connected. $\forall \langle a, \alpha \rangle, \langle b, \beta \rangle \in L$, a, b are real numbers and α, β are countable or finite ordinals, so $\langle a, \alpha \rangle$ and $\langle b, \beta \rangle$ can be connected by a path, as elements of $[0,1) \times (\alpha \cup \beta + 1)$. Therefore L is path-connected. □

By adding a greatest element ∞ to L, namely one-point compactification, we get the extended long ray $L^* = L + \{\infty\}$.

Proposition 3.1.3. L^* is connected but not path-connected.

Proof. L^* is connected because L is connected and the singleton $\{\infty\}$ is not open.

Suppose L^* is path-connected, then there exists a continuous function $f : [0,1] \to L^*, f(0) = \langle 0, 0 \rangle, f(1) = \infty$.

Now prove that f is a surjection. If not, there exists an element $p \in L^*$ so that $f^{-1}(\{p\}) = \varnothing$. Then $f^{-1}((\leftarrow, p))$ and $f^{-1}((p, \rightarrow))$ are disjoint open sets in $[0,1]$, and $f^{-1}((\leftarrow, p)) \cup f^{-1}((p, \rightarrow)) = [0,1]$. That contradicts that $[0,1]$ is connected.

Therefore for all $\alpha < \omega_1$, $f^{-1}((0,1) \times \{\alpha\})$ is nonempty. Since $(0,1) \times \{\alpha\} = (\langle 0, \alpha \rangle, \langle 0, \alpha + 1 \rangle)$ is open in L, $f^{-1}((0,1) \times \{\alpha\})$ is open in $[0, 1]$, so it must contain an open interval as a subset. That contradicts that there can't be uncountably many disjoint open intervals in $[0,1]$. □

There exists a continuous injection from $[0,1)$ to L, but couldn't exist a continuous surjection. Although L and $[0,1)$ have the same cardinality, L is much 'longer' than $[0,1)$. That is why we call it the long ray.

L is C_1 while L^* is not. Neither L nor L^* is C_2. That is quite similar with how we dealt with $[0, \omega_1)$ and $[0, \omega_1]$.

3.2 Tychonoff Plank

Definition 3.2.1. The *Tychonoff plank* T is defined as $[0, \omega_1] \times [0, \omega]$, here the topology is **not** the order topology of its canonical order, but the product topology.

By removing the element $\langle \omega_1, \omega \rangle$ of T, we get the deleted Tychonoff plank $T^- = T \setminus \{\langle \omega_1, \omega \rangle\}$.

Proposition 3.2.1. T is a compact, T_2 and is therefore T_4. However, T^- is not T_4. That shows a subspace of a T_4 space isn't necessarily T_4.

Proof. According to proposition 1.3.1 and proposition 2.3.2, both $[0, \omega]$ and $[0, \omega_1]$ are T_2 and compact. Because both T_2 and compactness are multiplicative, T is T_2 and compact and therefore T_4.

$[0, \omega) \times \{\omega\}$ and $\{\omega_1\} \times [0, \omega)$ are both closed sets of T^-. Consider any neighborhood of $\{\omega_1\} \times [0, \omega)$ denoted by U, $\forall i < \omega, ([0, \omega_1) \times \{i\}) \cap U$ must contain $(\alpha_n, \omega_1) \times \{i\}$ as a subset. Let $\alpha = \bigcup_{i<\omega} \alpha_n$, then $\alpha < \omega_1$, $(\alpha, \omega_1) \times [0, \omega) \subseteq U$. But any open set V containing $\langle \alpha+1, \omega\rangle$ must contain $\{\alpha+1\} \times (n, \omega)$ as a subset, here n is some ordinal less than ω. Therefore $\langle \alpha+1, n+1\rangle \in U \cap V$, T^- is not T_4. □

Definition 3.2.2. Let α be a limit ordinal, an *α-indexed sequence* in X means a function from α to X.

An ordinal-indexed sequence is the generalization of a sequence. A common sequence is a ω-indexed sequence.

If X is a topological space, we say that an α-indexed sequence of elements of X converges to a limit x, when given any neighborhood U of x there is an ordinal $\beta < \alpha$ such that x_γ is in U for all $\gamma > \beta$.

Ordinal-indexed sequences are usually more powerful than common sequences. For example, in $[0, \omega_1]$, ω_1 is a limit point of $[0, \omega_1)$, but it is not a limit of any common sequence in ω_1, see proposition 2.3.4. However, it is the limit of the ω_1-indexed sequence mapping any ordinal less than ω_1 to itself.

The next counterexample shows that even ordinal-indexed sequences are not powerful enough to replace the concept of limit points.

Proposition 3.2.2. In T, the corner point $\langle \omega_1, \omega\rangle$ is a limit point of $[0, \omega_1) \times [0, \omega)$, but it is not a limit of any ordinal-indexed sequence in $[0, \omega_1) \times [0, \omega)$.

Proof. Let S denote $[0, \omega_1) \times [0, \omega)$. Suppose $\exists f : \pi \to S$ converges to $\langle \omega_1, \omega\rangle$, then $\forall \langle \rho, \sigma\rangle \in S$, there exists a least $\alpha < \pi$, so that $\forall \beta \geqslant \alpha, f(\beta) \in (\rho, \omega_1) \times (\sigma, \omega)$. Define $g : S \to \pi, \langle \rho, \sigma\rangle \mapsto \alpha$. Let's discuss some properties of g.

1. If $\rho_1 > \rho_0, \sigma_1 > \sigma_0$, then $g(\langle \rho_1, \sigma\rangle) \geqslant g(\langle \rho_0, \sigma\rangle)$ and $g(\langle \rho, \sigma_1\rangle) \geqslant g(\langle \rho, \sigma_0\rangle)$.
2. $g(\langle \rho, \sigma\rangle) = \max\{g(\langle \rho, 0\rangle), g(\langle 0, \sigma\rangle)\}$.

Define

$$g_0 : \omega \to \pi, \sigma \mapsto g(\langle 0, \sigma\rangle)$$
$$g_1 : \omega_1 \to \pi, \rho \mapsto g(\langle \rho, 0\rangle)$$

g_0 and g_1 actually determine g. Their images $g_0([0, \omega))$ and $g_1([0, \omega_1))$ are subsets of π, so they are well-ordered sets. In fact, the order type of $g_0([0, \omega))$ can't be greater than ω, since g_0 is non-strictly increasing; similarly, the order type of $g_1([0, \omega_1))$ can't be greater than ω_1.

Now I claim that the order type of $g_0([0, \omega))$ is precisely ω. Suppose not, its order type is a finite ordinal, $g_0([0, \omega))$ has a greatest element α. So $\exists \beta < \omega, g_0([\beta, \omega)) = \{\alpha\}$. Then the α-th element of the sequence couldn't actually exist, since $\bigcap_{\gamma \in [\beta, \omega)} [0, \omega_1) \times (\gamma, \omega) = \varnothing$. Similarly, the order type of $g_1([0, \omega_1))$ is precisely ω_1, or there will exist a countable ordinal β_1, so that $g_1([\beta_1, \omega_1))$ is a singleton.

Claim: $\forall \alpha_0 \in g_0([0,\omega)), \exists \alpha_1 \in g_1([0,\omega_1))$, so that $\alpha_0 < \alpha_1$; and $\forall \beta_1 \in g_1([0,\omega_1)), \exists \beta_0 \in g_0([0,\omega))$, so that $\beta_1 < \beta_0$. Suppose not, without loss of generality, assume $\exists \alpha \in g_0([0,\omega))$, α is greater than any element of $g_1([0,\omega_1))$. Then $g([0,\omega_1) \times g_0^{-1}(\alpha)) = \{\alpha\}$, so the α-th element of the sequence couldn't exist.

However, the last two assertions can't be true together. $\forall \alpha_i \in g_0([0,\omega)), \exists \beta_i \in g_1([0,\omega_1))$ so that $\alpha_i < \beta_i$, since the order type of $g_1([0,\omega_1))$ is ω_1, so $\forall i < \omega, [0,\beta_i) \cap g_1([0,\omega_1))$ is a countable or finite set. Then $\bigcup_i [0,\beta_i) \cap g_1([0,\omega_1))$ is a countable set, $\exists \beta \in g_1([0,\omega_1))$, so that $\forall i, \beta > \beta_i > \alpha_i$ since $g_1([0,\omega_1))$ is uncountable.

Therefore there can't be an ordinal-indexed sequence converging to $\langle \omega_1, \omega \rangle$.

$\square$

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